

Examples of interacting particle systems on \mathbb{Z} as Pfaffian point processes II - coalescing branching random walks and annihilating random walks with immigration.

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Abstract

Two classes of interacting particle systems on \mathbb{Z} are shown to be Pfaffian point processes at fixed times, and for all deterministic initial conditions. The first comprises coalescing and branching random walks, the second annihilating random walks with pairwise immigration. Various limiting Pfaffian point processes on \mathbb{R} are found by diffusive rescaling, including the point set process for the Brownian net.

1 Introduction and statement of key results

This paper continues the study in [3], where it was shown that systems of instantaneously annihilating or coalescing random walks on \mathbb{Z} are Pfaffian point processes, at any fixed time and for all deterministic initial conditions. The purpose in this paper is to describe two additional mechanisms that preserve this Pfaffian property. The Pfaffian property should be useful to investigate statistics, such as asymptotics of correlation functions (as in Theorem 1 of [13]) or for studying gap probabilities (as in [12], [9] or [8] for examples from random matrix ensembles).

The algebraic structure of the generators of various one dimensional interacting particle systems (without necessarily preservation of particle numbers) has been investigated, and is in many examples intimately linked to Hecke algebras, see the reviews [1] and [6]. We would like to better understand how to deduce from these algebraic properties the concrete statistical properties of our models, such as the correlation functions, and in particular what algebraic properties lie behind the emergence of the Pfaffian property.

1.1 (BCRW) Branching coalescing random walks.

In addition to instantaneously coalescing random walks, we allow nearest neighbour binary branching: any given particle may instantaneously produce an independent copy at a nearest neighbour. The dynamics of this continuous-time model on \mathbb{Z} are informally described as follows. Between interactions particles perform independent nearest neighbour random walks with jumps

$$x \rightarrow x - 1 \text{ at rate } q, \quad \text{and} \quad x \rightarrow x + 1 \text{ at rate } p.$$

If a particle jumps onto an occupied site then the two particles instantaneously coalesce. Independently a particle branches

$$x \rightarrow \{x, x - 1\} \text{ at rate } \ell, \quad \text{and} \quad x \rightarrow \{x, x + 1\} \text{ at rate } r.$$

Branching events respect coalescence: if a particle branches onto an occupied site then the existing and new particles instantaneously coalesce.

The generator is given in (10), and this characterises the law of a process with values in $\{0, 1\}^{\mathbb{Z}}$. We write $(\eta_t : t \geq 0)$ for canonical variables and \mathbb{P}_η for the law corresponding to an initial condition $\eta \in \{0, 1\}^{\mathbb{Z}}$. As in [3], we consider η_t as a point process on \mathbb{Z} , and our aim is to establish that this is a Pfaffian point process for a suitable Pfaffian kernel $\mathbf{K}(x, y)$, which we now describe.

Recall the notation from [3]: for $\eta \in \{0, 1\}^{\mathbb{Z}}$ and $y \leq z$ we write

$$\eta[y, z] = \sum_{y \leq x < z} \eta(x)$$

so that $\eta[y, y] = 0$. For $f : \mathbb{Z} \rightarrow \mathbb{R}$ we write difference operators as

$$D^+ f(x) = f(x + 1) - f(x), \quad D^- f(x) = f(x - 1) - f(x).$$

Theorem 1 *Let $(\eta_t : t \geq 0)$ be the BCRW model with parameter values satisfying*

$$p\ell = qr, \quad p, q > 0. \tag{1}$$

For any initial condition $\eta \in \{0, 1\}^{\mathbb{Z}}$, and at any fixed time $t \geq 0$, the variable η_t is a Pfaffian point process with kernel \mathbf{K} given, for $y < z$, by

$$\mathbf{K}(y, z) = \frac{1}{\phi} \begin{pmatrix} K_t(y, z) & -D_2^+ K_t(y, z) \\ -D_1^+ K_t(y, z) & D_1^+ D_2^+ K_t(y, z) \end{pmatrix}, \tag{2}$$

and $\mathbf{K}_{12}(y, y) = 1 - \frac{1}{\phi} K_t(y, y + 1)$, with other entries determined by anti-symmetry, where

$$K_t(y, z) = \phi^{z-y} \mathbb{P}_\eta[\eta_t[y, z] = 0], \quad \text{for } t \geq 0 \text{ and } y \leq z,$$

and $\phi = \sqrt{1 + \frac{\ell}{q}} = \sqrt{1 + \frac{r}{p}}$. The same result holds when $p = r = 0$ and $q, \ell > 0$ taking $\phi = \sqrt{1 + \frac{\ell}{q}}$, or when $q = \ell = 0$ and $p, r > 0$ taking $\phi = \sqrt{1 + \frac{r}{p}}$.

Remarks. 1. For random initial conditions the law of η_t is not in general a Pfaffian point process, though by conditioning on the initial condition η_0 the correlation functions can always be expressed as the expectation of a Pfaffian with a random kernel $\mathbf{K}^{(\eta_0)}$ depending on η_0 :

$$\rho_t^{(n)}(x_1, \dots, x_n) = \mathbb{E} (\text{Pf} (\mathbf{K}^{\eta_0}(x_i, x_j) : i, j \leq n)). \quad (3)$$

For certain random initial conditions, including the natural case when the sites $(\eta_0(x) : x \in \mathbb{Z})$ are independent, the expectation can be taken inside the Pfaffian and the process does remain a Pfaffian point process (see the remarks after the proof of Lemma 6). Moreover the invariant measure, which is product Bernoulli, can be considered as a Pfaffian point process (see the remark at the end of section 2.)

2. The restriction $pl = qr$ seems to be necessary for the Pfaffian point process property, and we don't fully understand its origin. Luckily, in the large scale diffusive limits explained below this restriction plays no role. Indeed the continuum systems depend on three parameters, for example two controlling the diffusion and drift rates and one the branching rate, and all these models are Pfaffian point processes. For the four parameter lattice models, there exist duality functions even when the restriction $pl = qr$ is not true, and it would be of interest to examine whether some other algebraic structure holds for the correlation functions.

3. In section 4 we investigate continuum limits to our Pfaffian point kernels, under space time diffusive scaling, yielding Pfaffian point processes on \mathbb{R} . For the branching parameter to have an effect in the continuum kernel, it needs to be scaled to grow suitably fast. This however is well understood in the construction of the Brownian net (see Sun and Swart [11]), and we of course need the same scaling of the branching as in the discrete time branching random walk approximations to the net. The Brownian net is a continuum collection of space time paths found by scaling discrete time branching coalescing random walks started at all space-time lattice points. The point set process $(\xi_t^A : t \geq 0)$ within the net, is defined by looking at all points at time t that are on paths that start at time zero in a set $A \subset \mathbb{R}$. It is known to be a Feller process taking values in the compact sets of $\overline{\mathbb{R}}$ with a suitable Hausdorff metric (see Theorem 1.11 in [11]). The approximating discrete time branching coalescing random walks used in the construction of the net are not Pfaffian point processes (the discrete time difference equations analogous to the Markov duality below are not solved by Pfaffians). However the difference between the discrete time models and continuous time models does not affect the continuum limits. We leave the verification of these technical details to a forthcoming paper, but we state here the resulting Pfaffian property for the Brownian net point set process, which answers the first open problem in section 8.3 of the survey paper on the net [10]. The Brownian net can have a parameter $b > 0$, controlling the branching rate, where the embedded left-right paths have drifts $\pm b$. The standard Brownian net corresponds to $b = 1$.

Proposition 2 *The transition density $p_t(A, dB)$ for the Brownian net point set process (ξ_t^A) is given by the law of the closed support of a Pfaffian point process on $\overline{\mathbb{R}}$ with the kernel \mathbf{K}_t^A*

of the form

$$\begin{aligned} \mathbf{K}_t^A(y, z) &= \begin{pmatrix} K_t^A(y, z) & -D_2 K_t^A(y, z) \\ -D_1 K_t^A(y, z) & D_1 D_2 K_t^A(y, z) \end{pmatrix} \quad \text{for } y < z, \\ (\mathbf{K}_t^A)_{12}(y, y) &= -D_2 K_t^A(y, y) + b \end{aligned} \quad (4)$$

where $(K_t^A(y, z) : y, z \in \mathbb{R}^2 : y \leq z)$ is the unique bounded solution to the PDE

$$\begin{cases} \partial_t K_t^A(y, z) &= \frac{1}{2} \Delta K_t^A(y, z) - b^2 K_t^A(y, z) \\ K_t^A(y, y) &= 1 \\ K_0^A(y, z) &= \mathbf{I}((y, z) \cap A = \emptyset). \end{cases} \quad (5)$$

The examples in section 4 will illustrate how continuum kernels arise of this form.

1.2 (ARWPI) Annihilating random walks with pairwise immigration.

As for the BCRW model, particles jump left or right at rates q and p . If a particle jumps onto an occupied site then the two particles instantaneously annihilate. In addition, independently there is

immigration of a pair of particles on sites $\{x, x-1\}$ at rate m .

Immigration respects annihilation: if a particle immigrates onto an occupied site then the existing and new particles instantaneously annihilate. The generator is given in (12).

Theorem 3 *For any initial condition $\eta \in \{0, 1\}^{\mathbb{Z}}$, and at any fixed time $t \geq 0$, the variable η_t^A is a Pfaffian point process with kernel \mathbf{K} given, for $y < z$, by*

$$\mathbf{K}(y, z) = \frac{1}{2} \begin{pmatrix} K_t(y, z) & -D_2^+ K_t(y, z) \\ -D_1^+ K_t(y, z) & D_1^+ D_2^+ K_t(y, z) \end{pmatrix}, \quad (6)$$

and $\mathbf{K}_{12}(y, y) = -\frac{1}{2} D_2^+ K_t(y, y)$, with other entries determined by anti-symmetry, where

$$K_t(y, z) = \mathbb{E}_\eta[(-1)^{n[y, z]}], \quad \text{for } t \geq 0 \text{ and } y \leq z.$$

Remarks. 1. The Glauber spin chain on \mathbb{Z} is an assignment of ± 1 spin values to each site which independently flip according to rates determined by nearest neighbour spins [5]. Sites favour aligned spin and at zero temperature a site surrounded by spins of the same sign does not flip, and the domain wall between regions of constant spin form a system of annihilating random walks on the dual lattice. At positive temperature, a spin may spontaneously flip regardless of its neighbours, and this corresponds to the creation of a pair of neighbouring domain walls. Since the Glauber model can be solved at all temperatures by mapping to a system of free fermions (Felderhof [2]), it is reasonable that the extra immigration of pairs does not destroy the Pfaffian property of solutions. A model with Poisson immigration of single particles is perhaps of more interest, but we do not see a simple algebraic structure behind this model.

2. We give in section 3 a spatially inhomogeneous version of Theorem 3, where the parameters p_x, q_x, m_x may be site dependent, in particular allowing immigration of particles at different rates at different places. Continuum limits can also be found for the ARWPI model under diffusive rescaling, where the immigration rate m must be scaled suitably so that they have a non-trivial effect in the limit. It is a pleasant fact that in many cases the Pfaffian kernel can be found completely explicitly. For example in example (d) in section 4, which we call the Brownian firework, pairs of particles are immigrated only at the origin at an infinite rate. This has a steady state $X_\infty^{(c)}$ where the immigration and the annihilation balance each other: $X_\infty^{(c)}$ is a Pfaffian point process on $\mathbb{R} \setminus \{0\}$ with kernel $\mathbf{K}_\infty^{(c)}$ of the form

$$\mathbf{K}_\infty^{(c)}(y, z) = \frac{1}{2} \begin{pmatrix} K_\infty^{(c)}(y, z) & -D_2 K_\infty^{(c)}(y, z) \\ -D_1 K_\infty^{(c)}(y, z) & D_1 D_2 K_\infty^{(c)}(y, z) \end{pmatrix} \quad \text{for } y < z,$$

$$(\mathbf{K}_\infty^{(c)})_{12}(y, y) = -\frac{1}{2} D_2 K_\infty^{(c)}(y, y)$$

where

$$K_\infty^{(c)}(y, z) = \begin{cases} 1 + \frac{2}{\pi} \left(\arctan \frac{y}{z} - \arctan \frac{z}{y} \right) & \text{when } 0 < y < z, \\ 0 & \text{when } y < 0 < z, \\ 1 + \frac{2}{\pi} \left(\arctan \frac{z}{y} - \arctan \frac{y}{z} \right) & \text{when } y < z < 0, \end{cases} \quad (7)$$

The corresponding intensity is given by $\rho_\infty^{(1)}(y) = \frac{1}{\pi|y|}$. Moreover since $\mathbf{K}_\infty^{(c)}(y, z) = 0$ when $y < 0 < z$ it is simple to deduce that

$$\rho_\infty^{(n+m)}(y_1, \dots, y_n, z_1, \dots, z_m) = \rho_\infty^{(n)}(y_1, \dots, y_n) \rho_\infty^{(m)}(z_1, \dots, z_m)$$

when $y_1, \dots, y_n < 0 < z_1, \dots, z_m$ and hence that $X_\infty^{(c)}|_{(-\infty, 0)}$ and $X_\infty^{(c)}|_{(0, \infty)}$ are independent point processes. The infinite strength firework of particles at the origin leads to the two half spaces being independent.

3. In theory all information about the process (at a fixed t) is contained in the single function $K_t(y, z)$, but extracting useful information remains of great interest (consider for example the efforts studying the Fredholm Pfaffians for gap probabilities for other Pfaffian point processes). One simple consequence of the Pfaffian structure is an estimate showing exponential convergence to equilibrium (which is a product Bernoulli distribution). Indeed we claim that there exist C_N for all $N \geq 1$ so that

$$\left| \rho_t^{(N)}(y_1, \dots, y_N) - \rho_\infty^{(N)}(y_1, \dots, y_N) \right| \leq C_N e^{-2mt} \quad (8)$$

uniformly over for all y_1, \dots, y_N and all initial conditions. (Recall that m is the immigration rate of pairs). This follows for a deterministic initial condition η once we show that

$$|K_t(y, z) - K_\infty(y, z)| \leq 2e^{-2mt} \quad \text{for all } y, z \in \mathbb{Z}, \quad (9)$$

since the entries in the kernel \mathbf{K} are differences of the bounded function K_t , so that the Pfaffian formula for $\rho_t^{(N)}$ is given by a finite linear combination of finite products of $K_t(y_i, y_j)$. For a general initial condition, one can first condition on the initial condition as in (3) and then use the fact that the above estimates are uniform in η .

To show (9) we can solve for $K_t(y, z)$ explicitly. Indeed, fixing a deterministic initial condition η , the kernel $K_t(y, z)$ has a representation in terms of a pair of independent continuous time random walkers (Y_t, Z_t) with generator $qD^+ + pD^-$, started at $Y_0 = y, Z_0 = z$. Let $\tau = \inf\{t : Y_t = Z_t\}$. Then the equation solved by $K_t(y, z)$ (see Lemma 7) shows that

$$K_t(y, z) = \mathbb{E} [e^{-2m\tau} \mathbf{I}(\tau \leq t)] + e^{-2mt} \mathbb{E} [K_0(Y_t, Z_t) \mathbf{I}(\tau > t)]$$

where $K_0(y, z) = (-1)^{\eta[y, z]}$ is bounded by 1. The limit $K_\infty(y, z) = \mathbb{E}[e^{-2m\tau}]$ and the estimate (9) follows easily from subtracting these two probabilistic representations. Solving explicitly we have

$$K_\infty(y, z) = \theta^{z-y}, \quad \text{where } \theta \in (0, 1) \text{ solves } \theta + \theta^{-1} - 2 = \frac{2m}{p+q}.$$

The Pfaffian kernel of the form (6) corresponding to $K_\infty(y, z)$ is

$$\mathbf{K}(y, z) = \frac{\theta^{z-y}}{2} \begin{pmatrix} 1 & (1-\theta) \\ (1-\theta^{-1}) & (1-\theta)(1-\theta^{-1}) \end{pmatrix},$$

and $\mathbf{K}_{12}(y, y) = (1-\theta)/2$. A little manipulation shows that this is a kernel for a product Bernoulli($\hat{\theta}$) distribution, where

$$\hat{\theta} = \frac{1-\theta}{2} = \frac{1}{2} \left(\sqrt{\frac{m^2}{(p+q)^2} + \frac{2m}{p+q}} - \frac{m}{p+q} \right)$$

Indeed by conjugating with elementary row and column operations (which leaves the corresponding point process unaltered) $\mathbf{K}(y, z)$ for $y < z$ can be changed successively to

$$\mathbf{K}(y, z) \rightarrow \frac{\theta^{z-y}}{2} \begin{pmatrix} 1 & 0 \\ \theta - \theta^{-1} & 0 \end{pmatrix} \rightarrow \frac{\theta^{z-y}}{2} \begin{pmatrix} 0 & 0 \\ \theta - \theta^{-1} & 0 \end{pmatrix},$$

while leaving $\mathbf{K}_{12}(y, y) = (1-\theta)/2$ unchanged. Then the Pfaffian for $\rho^{(N)}$ has only a single non-zero entry on the top row, and expanding along this row one finds

$$\rho^{(N)}(y_1, \dots, y_N) = \frac{1-\theta}{2} \rho^{(N-1)}(y_2, \dots, y_N).$$

We remark that no exponential convergence statement such as (8) holds for the BCRW model, since empty gaps in the initial condition can only be filled at linear speed. However, for many initial conditions there is weak convergence to a Bernoulli invariant measure - see the remarks in section 2.

2 Proof of Theorem 1.

We start with a terse summary of the main steps. The proof in [3] for coalescing systems without branching uses the empty interval duality function; this function remains a duality function for the branching model, but the empty interval probabilities are no longer given by a Pfaffian; however the duality function can be adjusted (by a suitable phase factor) in a way that again yields Pfaffians. The use of empty interval probabilities to study branching

systems is not new, see for example Krebs et al. [7] where the equations for a single empty interval are used to study various finite systems.

The generator for BCRW is given, for suitable $F : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$, by

$$\begin{aligned} \mathcal{L}F(\eta) = & q \sum_{x \in \mathbb{Z}} (F(\eta_{x,x-1}) - F(\eta)) + p \sum_{x \in \mathbb{Z}} (F(\eta_{x-1,x}) - F(\eta)) \\ & + \ell \sum_{x \in \mathbb{Z}} (F(\eta_{x,x-1}^b) - F(\eta)) + r \sum_{x \in \mathbb{Z}} (F(\eta_{x,x+1}^b) - F(\eta)), \end{aligned} \quad (10)$$

where $\eta_{x,y}$ (resp. $\eta_{x,y}^b$) is the new configuration resulting from a jump (respectively a branch) from x onto y

$$\begin{cases} \eta_{x,y}(z) = \eta_{x,y}^b(z) = \eta(z) & \text{for } z \notin \{x, y\}, \\ \eta_{x,y}(x) = 0, & \eta_{x,y}^b(x) = \eta(x), \\ \eta_{x,y}(y) = \eta_{x,y}^b(y) = \min\{1, \eta(x) + \eta(y)\}. \end{cases}$$

For $n \geq 1$ and $y = (y_1, \dots, y_{2n})$ with $y_1 \leq \dots \leq y_{2n}$ we define the function $\Sigma_y(\eta)$ as the indicator that the intervals $[y_1, y_2), \dots, [y_{2n-1}, y_{2n})$ are all empty; explicitly

$$\Sigma_y(\eta) = \prod_{i=1}^n \mathbf{I}(\eta[y_{2i-1}, y_{2i}) = 0).$$

We define a one-particle operator, acting on $f : \mathbb{Z} \rightarrow \mathbb{R}$, by

$$L^{p,q}f(x) = qD^+f(x) + pD^-f(x).$$

Lemma 4 *For $y_1 < \dots < y_{2n}$ the action of the generator \mathcal{L} on $\Sigma_y(\eta)$ is given by*

$$\mathcal{L}\Sigma_y(\eta) = \sum_{i=1}^n (L_{y_{2i-1}}^{p+r,q} + L_{y_{2i}}^{p,q+l}) \Sigma_y(\eta)$$

where the subscript y_i indicates the variable upon which the operator acts.

Proof of Lemma 4. A direct check shows that the terms of \mathcal{L} coming from left and right jumping contribute

$$q \sum_{x \in \mathbb{Z}} (\Sigma_y(\eta_{x,x-1}) - \Sigma_y(\eta)) + p \sum_{x \in \mathbb{Z}} (\Sigma_y(\eta_{x-1,x}) - \Sigma_y(\eta)) = \sum_{i=1}^{2n} L_{y_i}^{p,q} \Sigma_y(\eta)$$

to $\mathcal{L}\Sigma_y(\eta)$ (see [3] for the details of this calculation). It remains to compute the terms arising from branching. Consider the term from left branching. The modified branching configuration $\eta_{x,x-1}^b$ differs from η only at the site $x-1$, so for each x there can be a change in at most one of the indicators in Σ_y , so we may write

$$\begin{aligned} & \ell (\Sigma_y(\eta_{x,x-1}^b) - \Sigma_y(\eta)) \\ = & \ell \sum_{i=1}^n \left(\prod_{j=1, j \neq i}^n \mathbf{I}(\eta[y_{2j-1}, y_{2j}) = 0) \right) \left(\mathbf{I}(\eta_{x,x-1}^b[y_{2i-1}, y_{2i}) = 0) - \mathbf{I}(\eta[y_{2i-1}, y_{2i}) = 0) \right). \end{aligned}$$

Fix $y < z$ and consider the generator contribution for a single empty interval indicator, namely

$$\ell \sum_{x \in \mathbb{Z}} \left(\mathbf{I}(\eta_{x,x-1}^b[y, z] = 0) - \mathbf{I}(\eta[y, z] = 0) \right).$$

The terms indexed by $x \leq y$ and $x \geq z+1$ are zero, as the modified configuration is unchanged in the interval $[y, z]$. The terms indexed by $y \leq x \leq z-1$ are also zero since there must be a particle at x to branch to the left from, in which case both empty interval indicators are zero. The remaining summand, when $x = z$, is given by

$$\begin{aligned} \ell \mathbf{I}(\eta_{z,z-1}^b[y, z] = 0) - \ell \mathbf{I}(\eta[y, z] = 0) &= \ell ((1 - \eta(z)) - 1) \mathbf{I}(\eta[y, z] = 0) \\ &= \ell D_z^+ \mathbf{I}(\eta[y, z] = 0). \end{aligned}$$

A similar calculation reveals that the term of the generator arising from right branching satisfies

$$r \sum_{x \in \mathbb{Z}} \left(\mathbf{I}(\eta_{x,x+1}^b[y, z] = 0) - \mathbf{I}(\eta[y, z] = 0) \right) = r D_y^- \mathbf{I}(\eta[y, z] = 0).$$

Collecting up contributions gives the claimed action. ■

The expression for $\mathcal{L}\Sigma_y(\eta)$ in Lemma 4 has different operators acting on even co-ordinates y_{2i} and odd co-ordinates y_{2i-1} . The proof of the Pfaffian property in Lemma 6 below is facilitated if each co-ordinate has the same operator acting on it, and the aim is to introduce a suitable phase factor precisely to have this effect. The phase factor is defined by

$$\Phi(y) = \prod_{i=1}^n \phi^{(y_{2i} - y_{2i-1})} \quad \text{for } y = (y_1, \dots, y_{2n}) \in \mathbb{R}^{2n} \text{ and } n \geq 1$$

and the following lemma shows that the value $\phi = \sqrt{1 + \frac{\ell}{q}} = \sqrt{1 + \frac{r}{p}}$ is the correct choice.

Lemma 5 *Suppose the rates p, q, r, l satisfy (1) and ϕ is chosen as in the statement of Theorem 1. Then for $y_1 < \dots < y_{2n}$*

$$\Phi(y) \mathcal{L}(\Sigma_y)(\eta) = \sum_{i=1}^{2n} (L_{y_i}^{p\phi, q\phi} - c_0) (\Phi(y) \Sigma_y(\eta))$$

where $c_0 = \frac{1}{2}(r + l) - (p + q)(\phi - 1) = \frac{p+q}{2}(\phi - 1)^2 \geq 0$.

Proof. This is a direct calculation. For a function of one variable we find that a change of $f : \mathbb{Z} \rightarrow \mathbb{R}$ to $\tilde{f}(y) = c^y f(y)$, for $c \neq 0$, produces the change

$$L^{p,q} \tilde{f}(y) = L^{pc^{-1}, qc} \tilde{f}(y) + (pc^{-1} + qc - p - q) \tilde{f}(y).$$

We apply this in the even coordinates with $c = \phi$ and in the odd co-ordinates with $c = \phi^{-1}$. The value of ϕ is chosen so that the corresponding difference operators now agree on both sets of coordinates. Different potential terms are created at odd or even co-ordinates, but these can be summed and then redistributed equally between all co-ordinates, which yields the constant c_0 . The equivalent expressions for c_0 follow from the definition of ϕ . ■

Lemma 6 For all $\eta \in \{0, 1\}^{\mathbb{Z}}$, for all $n \geq 1$, $y_1 \leq \dots \leq y_{2n}$ and $t \geq 0$

$$\Phi(y) \mathbb{E}_\eta [\Sigma_y(\eta_t)] = \text{Pf}(K^{(2n)}(t, y)),$$

where $K^{(2n)}(t, y)$ is the anti-symmetric $2n \times 2n$ matrix with entries $K_t(y_i, y_j)$ defined, for $i < j$, by

$$K_t(y, z) = \phi^{z-y} \mathbb{P}_\eta[\eta_t[y, z] = 0], \quad \text{for } t \geq 0 \text{ and } y \leq z.$$

Proof of Lemma 6. We follow closely the arguments for the pure coalescing case, as in Lemma 3 of [3], and use the notation there for the cells V_{2n} and parts of the boundary $\partial V_{2n}^{(i)}$. We point out here only the changes needed.

To establish the identity, one checks that both sides are solutions to the system of ODEs, in this case

$$(ODE)_{2n} \quad \begin{cases} \partial_t u^{(2n)}(t, y) &= \sum_{i=1}^{2n} (L_{y_i}^{p\phi, q\phi} - c_0) u^{(2n)}(t, y) & \text{on } [0, \infty) \times V_{2n}, \\ u^{(2n)}(t, y) &= u^{(2n-2)}(t, y^{i, i+1}) & \text{on } [0, \infty) \times \partial V_{2n}^{(i)}, \\ u^{(2n)}(0, y) &= \Phi(y) \Sigma_y(\eta) & \text{on } V_{2n}, \end{cases}$$

taking $u^{(0)} \equiv 1$. This infinite system can be shown by induction on n to have unique solutions, within the class of functions that suitable exponential growth at infinity. As in [3], that $(t, y) \mapsto \mathbb{E}_\eta [\Phi(y) \Sigma_y(\eta_t)]$ is a solution follows from Lemma 5 and the extra fact that the phase factor satisfies $\Phi(y) = \Phi(y^{i, i+1})$ on the boundary $\partial V_{2n}^{(i)}$.

The fact that the Pfaffian is also the solution to this system follows as in the non-branching case in [3], with the only change being that we need to verify the extra phase term does not affect the initial condition being satisfied. However we may rewrite the entries in the Pfaffian at time zero using

$$K_0(y, z) = \phi^{z-y} \mathbf{I}(\eta[y, z] = 0) = \lim_{\theta \downarrow 0} \frac{(-\theta)^{\eta[a, z]} \phi^{z-a}}{(-\theta)^{\eta[a, y]} \phi^{y-a}} \quad \text{for } a < y < z.$$

The Pfaffian $\text{Pf}(K^{(2n)}(0, y))$ is therefore the limit of Pfaffians of a matrix A with entries in quotient form $A_{ij} = a_i/a_j$ for $i < j$. For such matrices one has $\text{Pf}(A) = (a_2 a_4 \dots)/(a_1 a_3 \dots)$ (see the appendix of [3] for example) and hence, taking $a < \min\{y_i\}$,

$$\text{Pf}(K^{(2n)}(0, y)) = \lim_{\theta \downarrow 0} \prod_{i=1}^n \frac{(-\theta)^{\eta[a, y_{2i}]} \phi^{y_{2i}-a}}{(-\theta)^{\eta[a, y_{2i-1}]} \phi^{y_{2i-1}-a}} = \Phi(y) \Sigma_y(\eta),$$

as required. ■

Remark. The last lemma is the point at which to observe that for certain random initial conditions, the Pfaffian property is still true. Indeed suppose that η_0 is random but that $\mathbb{E} [\Phi(y) \Sigma_y(\eta_0)]$ is still given by a $2n \times 2n$ Pfaffian with entries $K_0(y_i, y_j)$ for $i < j$, for some K_0 of exponential growth. The statement of the lemma then still holds, and so does Theorem 1, which is deduced from the lemma without any changes. A simple example is when $\eta_0(x)$ are independent Bernoulli(θ_x) variables. Then the condition above is true with

$$K_0(y, z) = \phi^{z-y} \prod_{k \in [y, z]} (1 - \theta_k).$$

A similar observation holds for the AWRPI model discussed in the next section.

Proof of Theorem 1. As in [3], the desired particle intensities $\mathbb{E}_\eta [\eta_t(x_1) \dots \eta_t(x_n)]$ may be recovered from the empty interval probabilities. From Lemma 6 we have

$$\mathbb{E}_\eta [\Sigma_y(\eta_t)] = \Phi(y)^{-1} \text{Pf}(K^{(2n)}(t, y)).$$

The factor $\Phi^{-1}(y)$ can be expressed as the determinant of a diagonal matrix $D(y)$ with entries $D_{ii}(y) = \phi^{(-1)^i y_i}$ for $i = 1, \dots, 2n$. The empty interval probabilities can then be expressed as a single Pfaffian

$$\mathbb{E}_\eta [\Sigma_y(\eta_t)] = \text{Pf}(D^{1/2}(y)K^{(2n)}(t, y)D^{1/2}(y)). \quad (11)$$

Note the ij 'th entry of the matrix $D^{1/2}(y)K^{(2n)}(t, y)D^{1/2}(y)$ is still a function only of the variables y_i, y_j . We now follow the argument in [3], where the intensities are derived from the empty interval probabilities via discrete derivatives. This leads to the process η_t being a Pfaffian point process with a kernel $\hat{\mathbf{K}}(y, z)$ where for $y < z$

$$\begin{pmatrix} \phi^{y+z} K_t(y, z) & -D_z^+ (\phi^{y-z} K_t(y, z)) \\ -D_y^+ (\phi^{z-y} K_t(y, z)) & D_y^+ D_z^+ (\phi^{-y-z} K_t(y, z)) \end{pmatrix},$$

and

$$\hat{\mathbf{K}}_{12}(y, y) = -D_z^+ (\phi^{y-z} K_t(y, z))|_{z=y} = 1 - \phi^{-1} K_t(y, y+1).$$

It remains to massage this kernel $\hat{\mathbf{K}}$ into the form \mathbf{K} stated in the Theorem, which uses only row and column operations that can be represented by conjugation with suitable matrices, that is we may replace $\hat{\mathbf{K}}(y, z)$ by $A(y)\hat{\mathbf{K}}(y, z)A^T(z)$ for any 2-by-2 matrix $A(y)$ (depending only on the variable y) that has determinant 1.

Expanding out the discrete derivatives in $\hat{\mathbf{K}}$ using the discrete product rule, and then conjugating the final matrix with a diagonal matrix $A(y) = \begin{pmatrix} \phi^{-y} & 0 \\ 0 & \phi^y \end{pmatrix}$ leads to an equivalent kernel, which we still denote $\hat{\mathbf{K}}$, with entries

$$\begin{aligned} \hat{\mathbf{K}}_{11}(y, z) &= K_t(y, z); \\ \hat{\mathbf{K}}_{12}(y, z) &= -(\phi^{-1} K_t(y, z+1) - K_t(y, z)); \\ \hat{\mathbf{K}}_{21}(y, z) &= -(\phi^{-1} K_t(y+1, z) - K_t(y, z)); \\ \hat{\mathbf{K}}_{22}(y, z) &= \phi^{-2} K_t(y+1, z+1) - \phi^{-1} K_t(y, z+1) - \phi^{-1} K_t(y+1, z) + K_t(y, z), \end{aligned}$$

$$\hat{\mathbf{K}}_{12}(y, y) = 1 - \phi^{-1} K_t(y, y+1).$$

Subtracting the first row and column from the second row and column, and then further conjugating with a diagonal matrix $A(y) = \begin{pmatrix} \phi^{-1/2} & 0 \\ 0 & \phi^{1/2} \end{pmatrix}$ gives the equivalent kernel \mathbf{K}

$$\hat{\mathbf{K}}(y, z) = \phi^{-1} \begin{pmatrix} K_t(y, z) & -K_t(y, z+1) \\ -K_t(y+1, z) & K_t(y+1, z+1) \end{pmatrix},$$

with $\hat{\mathbf{K}}_{12}(y, y) = 1 - \phi^{-1} K_t(y, y+1)$. Finally, the desired kernel \mathbf{K} is obtained by again subtracting the first row and column from the second. ■

Remarks. 1. Letting $t \rightarrow \infty$ the process, for suitable non-zero initial conditions, converges to an invariant Bernoulli distribution. It is fun to see this via the Pfaffian kernels by solving for $K_t(y, z)$ explicitly. Take the maximal initial condition $\eta_0(x) = 1$ for all x . Then $K_0(y, z) = 0$. and the kernel $K_t(y, z)$ has a representation in terms of a pair of independent continuous time random walkers (Y_t, Z_t) with generator $q\phi D^+ + p\phi D^-$, started at $Y_0 = y, Z_0 = z$. Let $\tau = \inf\{t : Y_t = Z_t\}$. The equation solved by $K_t(y, z)$ (from Lemma 5) shows that

$$K_t(y, z) = \mathbb{E} [e^{-2c_0\tau} \mathbf{I}(\tau \leq t)] \uparrow K_\infty(y, z) = \mathbb{E}[e^{-2c_0\tau}].$$

Solving explicitly we find $K_\infty(y, z) = \phi^{-(z-y)}$, and the Pfaffian kernel of the form (2) corresponding to $K_\infty(y, z)$ is

$$\mathbf{K}(y, z) = \frac{\phi^{y-z-1}}{2} \begin{pmatrix} 1 & (1 - \phi^{-1}) \\ (1 - \phi) & (1 - \phi)(1 - \phi^{-1}) \end{pmatrix},$$

and $\mathbf{K}_{12}(y, y) = 1 - \phi^{-2}$. A little manipulation, using row and column operations as for the ARWPI model, shows that this is a kernel for a product Bernoulli($1 - \phi^{-2}$) distribution. Convergence of $K_t(y, z)$ implies that all entries in the Pfaffian kernel converge, which in turn implies that the process η_t converges to product Bernoulli($1 - \phi^{-2}$) in distribution in the product topology.

For general non-zero initial conditions the same is almost true. Rather than analyse the kernel, we use a simple coupling argument for attractive nearest neighbour systems. All non zero solutions can be coupled between the maximal solution and a solution started from a single point. It therefore is enough convergence for the process η_t^0 started from a single occupied site, say the origin. But this process can be coupled with the process $\eta_t^{\mathbb{Z}}$ started from all occupied sites. Indeed by a graphical construction, or equivalently solving using the same Poisson drivers for jump and branch events, shows that

$$\eta_t^0(y) = \eta_t^{\mathbb{Z}}(y) \quad \text{for all } y \in [l_t, r_t],$$

where l_t, r_t mark the leftmost and rightmost occupied site in η_t^0 . The behaviour of the pair $\{l_t, r_t\}$ is however easy to understand: provided $p+l > q$ and $q+r > p$ we can ensure $l_t \rightarrow -\infty$ and $r_t \rightarrow \infty$. Under these conditions the process looks like $\eta_t^{\mathbb{Z}}$ in a growing interval, and we already know $\eta_t^{\mathbb{Z}}$ converges to Bernoulli equilibrium.

2. It is natural to look for a spatially inhomogeneous version of the BCRW model, where p_x, q_x, l_x, r_x are allowed to be site dependent. This was explored in the thesis [4] and the Pfaffian property can survive, but under a somewhat stronger condition on the parameters that we do not yet fully understand.

3 Proof of Theorem 3.

The result for the annihilating model with immigration follows by very similar lines, and we remark only on the changes caused by the new immigration term. The result holds for

systems with spatially inhomogeneous rates. There is no additional complexity in the proof, so we continue in this general framework.

The generator for (spatially inhomogeneous) ARWPI is given, for suitable $F : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}F(\eta) = & \sum_{x \in \mathbb{Z}} q_x (F(\eta_{x,x-1}) - F(\eta)) + \sum_{x \in \mathbb{Z}} p_x (F(\eta_{x-1,x}) - F(\eta)) \\ & + \sum_{x \in \mathbb{Z}} m_x (F(\eta_{x-1,x}^i) - F(\eta)), \end{aligned} \quad (12)$$

where $\eta_{x-1,x}^i$ is the new configuration resulting from a pair immigration onto $\{x-1, x\}$ defined by

$$\begin{cases} \eta_{x-1,x}^i(z) = \eta(z) & \text{for } z \notin \{x-1, x\}, \\ \eta_{x-1,x}^i(z) = 1 - \eta(z) & \text{for } z \in \{x-1, x\}. \end{cases}$$

Note that any immigrating particle instantly annihilates with any existing particle on the target site. We suppose m_x, p_x, q_x are uniformly bounded, so that this generator uniquely determines a Markov process.

The following spin product function is a Markov duality function for this generator (as used for the pure annihilating model in [3]). For $n \geq 1$ and $y = (y_1, \dots, y_{2n})$ with $y_1 \leq \dots \leq y_{2n}$ we define the spin product

$$\Sigma_y(\eta) = \prod_{i=1}^n (-1)^{\eta[y_{2i-1}, y_{2i}]}.$$

We define the one-particle operator L , acting on $f : \mathbb{Z} \rightarrow \mathbb{R}$, by

$$Lf(x) = q_x D^+ f(x) + p_x D^- f(x) - 2m_x f(x). \quad (13)$$

Lemma 7 *For $y_1 < \dots < y_{2n}$ the action of the generator \mathcal{L} on $\Sigma_y(\eta)$ is given by*

$$\mathcal{L}\Sigma_y(\eta) = \sum_{i=1}^{2n} L_{y_i} \Sigma_y(\eta).$$

Proof of Lemma 7. As in [3] the terms of \mathcal{L} coming from particle motion contribute

$$\sum_{x \in \mathbb{Z}} q_x (\Sigma_y(\eta_{x,x-1}) - \Sigma_y(\eta)) + \sum_{x \in \mathbb{Z}} p_x (\Sigma_y(\eta_{x-1,x}) - \Sigma_y(\eta)).$$

It remains to compute the immigration term. Note that the modified immigration configuration $\eta_{x-1,x}^i$ differs from η on at most two sites, $x-1$ and x . Since the y_i are strictly ordered, the intervals $[y_{2i-1}, y_{2i}]$ are separated by at least one site, whereby a pair of adjacent sites $-1, x$ can intersect at most one of the intervals. In particular any change due to immigration affects at most one interval $[y_{2i-1}, y_{2i}]$, leading to the formula

$$\Sigma_y(\eta_{x-1,x}^i) - \Sigma_y(\eta) = \sum_{i=1}^n \left(\prod_{j \neq i} (-1)^{\eta[y_{2j-1}, y_{2j}]} \right) \left((-1)^{\eta_{x-1,x}^i[y_{2i-1}, y_{2i}]} - (-1)^{\eta[y_{2i-1}, y_{2i}]} \right).$$

Fix $y < z$ and consider the generator contribution for a single spin product $(-1)^{\eta[y,z]}$, namely

$$\sum_{x \in \mathbb{Z}} m_x \left((-1)^{\eta_{x-1,x}^i[y,z]} - (-1)^{\eta[y,z]} \right).$$

The terms indexed by $x \leq y-1$ and $x \geq z+1$ are zero, as the modified configuration is unchanged in the interval $[y, z]$. The terms $y+1 \leq x \leq z-1$ are also zero, since the immigration of two particles does not change the parity of $\eta[y, z]$. The remaining terms give identical non-zero contributions: for $x = y$ or $x = z$

$$(-1)^{\eta_{x-1,x}^i[y,z]} - (-1)^{\eta[y,z]} = \prod_{\substack{w=y \\ w \neq x}}^{z-1} (-1)^{\eta(w)} \left((-1)^{1-\eta(x)} - (-1)^{\eta(x)} \right) = -2(-1)^{\eta[y,z]}.$$

All together the immigration term is given by

$$\sum_{x \in \mathbb{Z}} m_x (\Sigma_y(\eta_{x-1,x}^i) - \Sigma_y(\eta)) = -2\Sigma_y(\eta) \sum_{i=1}^{2n} m_{y_i}.$$

Collecting the jump and immigration terms gives the desired expression. ■

Proof of Theorem 3. Following the argument from [3], we first claim that for all $\eta \in \{0, 1\}^{\mathbb{Z}}$, for all $n \geq 1$, $y_1 \leq \dots \leq y_{2n}$ and $t \geq 0$

$$\mathbb{E}_\eta [\Sigma_y(\eta_t)] = \text{Pf}(K^{(2n)}(t, y)),$$

where $K^{(2n)}(t, y)$ is the anti-symmetric $2n \times 2n$ matrix with entries $K_t(y_i, y_j)$ for $i < j$, defined by $K_t(y, z) = \mathbb{E}_\eta[(-1)^{\eta[y,z]}]$. The particle intensities $\mathbb{E}_\eta[\eta_t(x_1) \dots \eta_t(x_n)]$ can then be recovered from product spin expectations via discrete derivatives and yield the stated kernel $\mathbf{K}(y, z)$. ■

4 Some continuum Pfaffian point process limits.

The entries for the Pfaffian kernels $\mathbf{K}(x, y)$ in both the branching model and the immigration model, are determined by a scalar function $K_t(x, y)$ that solves a certain discrete heat equation. Under diffusive space time scaling, and with suitable scaling of the parameters, we can obtain natural limiting Pfaffian point processes on \mathbb{R} . We record here certain examples, simply to add to the rather small current list of explicit Pfaffian point process kernels. Two points are perhaps of greatest interest:

1. Unlike the continuum examples from [3], alongside the diffusive scaling of space-time, the reaction parameters controlling branching and immigration must be simultaneously scaled, so that they have a non-trivial effect on the continuum limit. Indeed branching but instantly coalescing Brownian motions do not have a simple meaning, and nor does immigration of instantly coalescing pair of Brownian motions onto the same point. However, since both discrete process are Pfaffian whose entire statistics are controlled by a kernel whose entries

solve a discrete PDE, the correct scaling for the parameters is easily revealed by examining the convergence for the differential equations.

2. The discrete equations behind coalescing models with branching and annihilating models with pairwise immigration are both discrete heat equations with a constant potential. This can be used to show there is an equality in law for the fixed time particle positions between these two models, if parameter values and initial values are chosen carefully. This connection exists at the discrete level (see [4]) but is most transparent for the limiting continuum models, and we detail this in the remarks after example (b).

Examples of continuum kernels.

In each of the four examples below we define

$$X_t^{(\epsilon)}(dx) = \eta_{\epsilon^{-2}t}(\epsilon^{-1}dx) \quad \text{on } \epsilon\mathbb{Z}$$

where η_t is one of the models studied earlier, with an initial condition and ϵ dependent parameters which we will specify. The point process $X_t^{(\epsilon)}$ will be a Pfaffian point process on $\epsilon\mathbb{Z}$ with a kernel $\mathbf{K}_t^{(\epsilon)}$. The diffusive scaling above is chosen so that an isolated non-interacting particle will converge to a Brownian motion. The entries of $\mathbf{K}_t^{(\epsilon)}$ are in terms of a scalar function $K_t^{(\epsilon)}(y, z)$ that will solve a lattice PDE that naturally scales to a continuum PDE. This will allow us to show the convergence of associated Point processes. We claim convergence of the particle system only at a fixed time. Indeed, for all $t \geq 0$, in each of the examples (a),(b),(c) below we claim $X_t^{(\epsilon)} \rightarrow X_t^{(c)}$ in distribution, on the space of locally finite point measures on \mathbb{R} with the topology of vague convergence (for the final example (d) we restrict to a region away from the origin). Moreover the limit $X_t^{(c)}$ is a simple point process and a Pfaffian point process on \mathbb{R} . In our examples we can often solve explicitly for the limiting kernel $\mathbf{K}_t^{(c)}(x, y)$.

The proof of the convergence $X_t^{(\epsilon)} \rightarrow X_t^{(c)}$ follows from the suitable convergence of the scalar functions $K_t^{(\epsilon)}(y, z)$ and their discrete derivatives to the analogous solutions of a continuum PDE, plugging in to the kernel convergence lemma from [3]. However we omit the details verifying the conditions of this lemma.

(a) Annihilating model with constant pairwise immigration.

We consider the ARWPI model with parameters $p_x = q_x = \alpha > 0$ and $m_x = \beta\epsilon^{-2} \geq 0$ for all x , and with zero initial condition. From Theorem 3 the process $X_t^{(\epsilon)}$ is a Pfaffian point process on $\epsilon\mathbb{Z}$ with kernel $\mathbf{K}_t^{(\epsilon)}$ of the form

$$\begin{aligned} \mathbf{K}_t^{(\epsilon)}(y, z) &= \frac{\epsilon}{2} \begin{pmatrix} K_t^{(\epsilon)}(y, z) & -D_2^{(\epsilon)} K_t^{(\epsilon)}(y, z) \\ -D_1^{(\epsilon)} K_t^{(\epsilon)}(y, z) & D_1^{(\epsilon)} D_2^{(\epsilon)} K_t^{(\epsilon)}(y, z) \end{pmatrix} \quad \text{for } y < z, \\ (\mathbf{K}_t^{(\epsilon)})_{12}(y, y) &= -\frac{\epsilon}{2} D_2^{(\epsilon)} K_t^{(\epsilon)}(y, y) \end{aligned} \tag{14}$$

where $D_i^{(\epsilon)}$ is the right discrete derivative on $\epsilon\mathbb{Z}$ (that is $D^{(\epsilon)}f(x) = \epsilon^{-1}(f(x + \epsilon) - f(x))$)

acting on the i 'th variable. The function $K_t^{(\epsilon)}(y, z)$ solves, for $y, z \in \epsilon\mathbb{Z}$ with $y \leq z$,

$$\begin{cases} \partial_t K_t^{(\epsilon)} &= \alpha \Delta^{(\epsilon)} K_t^{(\epsilon)} - 2\beta K_t^{(\epsilon)}, \\ K_t^{(\epsilon)}(y, y) &= 1, \\ K_0^{(\epsilon)}(y, z) &= 1. \end{cases} \quad (15)$$

Here $\Delta^{(\epsilon)}$ is the discrete Laplacian on $(\epsilon\mathbb{Z})^2$.

The limit X_t is Pfaffian on \mathbb{R} with kernel of the form

$$\begin{aligned} \mathbf{K}_t^{(c)}(y, z) &= \frac{1}{2} \begin{pmatrix} K_t^{(c)}(y, z) & -D_2 K_t^{(c)}(y, z) \\ -D_1 K_t^{(c)}(y, z) & D_1 D_2 K_t^{(c)}(y, z) \end{pmatrix} \quad \text{for } y < z, \\ (\mathbf{K}_t^{(c)})_{12}(y, y) &= -\frac{1}{2} D_2 K_t^{(c)}(y, y) \end{aligned} \quad (16)$$

(D_i is the derivative in the i th co-ordinate) where $K_t^{(c)}(y, z)$ is C^2 on $\{y, z \in \mathbb{R}^2 : y \leq z\}$ and solves

$$\begin{cases} \partial_t K_t^{(c)}(y, z) &= \alpha \Delta K_t^{(c)}(y, z) - 2\beta K_t^{(c)}(y, z) \\ K_t^{(c)}(y, y) &= 1 \\ K_0^{(c)}(y, z) &= 1. \end{cases} \quad (17)$$

The unique bounded solution $K_t^{(c)}(y, z)$ has a probabilistic representation in terms of a two dimensional Brownian motion (Y_t, Z_t) , run at rate 2α and started at (y, z) , namely

$$K_t^{(c)}(y, z) = \mathbb{E} \left[e^{-2\beta(t \wedge \tau)} \right] \quad \text{where } \tau = \inf\{t : Y_t = Z_t\}.$$

Solving for $K_t^{(c)}(y, z)$ explicitly allows one to read off the one point density

$$\rho_t^{(1)}(y) = -\frac{1}{2} D_2 K_t^{(c)}(y, y) = \frac{1}{2} \sqrt{\frac{\beta}{\alpha}} \operatorname{erf}(\sqrt{2t\gamma}).$$

The kernel also has a limit as $t \rightarrow \infty$, in particular $K_t^{(c)}(y, z) \rightarrow K_\infty^{(c)}(y, z)$ where

$$K_\infty^{(c)}(y, z) = \mathbb{E}[e^{-2\beta\tau}] = e^{-\sqrt{\frac{\beta}{\alpha}}(z-y)}.$$

It is no longer enough in the continuum to just examine convergence of $K_t^{(c)}$, but an examination of the exact formula shows that both $K^{(c)}$ and its first two derivatives converge, uniformly over y, z , as $t \rightarrow \infty$, which implies that the continuum point processes $X_t^{(c)}$ converge as $t \rightarrow \infty$ to a point process $X_\infty^{(c)}$ (one can follow the steps of the proof of Lemma 4 from [3]). The limit has kernel

$$\mathbf{K}_\infty^{(c)}(y, z) = \frac{1}{2} \begin{pmatrix} e^{-\sqrt{\frac{\beta}{\alpha}}(z-y)} & \sqrt{\frac{\beta}{\alpha}} e^{-\sqrt{\frac{\beta}{\alpha}}(z-y)} \\ -\sqrt{\frac{\beta}{\alpha}} e^{-\sqrt{\frac{\beta}{\alpha}}(z-y)} & -\frac{\beta}{\alpha} e^{-\sqrt{\frac{\beta}{\alpha}}(z-y)} \end{pmatrix} \quad \text{for } y < z, \quad (18)$$

and $\mathbf{K}_{\infty,12}^{(c)}(y, y) = \frac{1}{2} \sqrt{\frac{\beta}{\alpha}}$. One can identify the limit, this time a disguised form for the kernel for a Poisson process. Indeed the same row and column operations as in the discrete

case allow the Pfaffian of the above kernel to be easily computed explicitly and the n -point intensity is given by

$$\rho_{\infty}^{(n)}(y_1, \dots, y_n) \equiv \left(\frac{1}{2} \sqrt{\frac{\beta}{a}} \right)^n.$$

Thus the distribution of the (continuum) point process in the large time limit is a Poisson process rate $\frac{1}{2} \sqrt{\frac{\beta}{a}}$. The four entries in the kernel converge exponentially to the $t = \infty$ limit. As in the discrete ARWPI model, this can be used to show the exponentially fast convergence $\rho_t^{(n)}(y_1, \dots, y_n) \rightarrow \rho_{\infty}^{(n)}(y_1, \dots, y_n)$ as $t \rightarrow \infty$, for any fixed n and uniformly over y_i .

Explicit formulae can be found for a variety of other initial conditions (see [4]). For example for an initial Bernoulli($\epsilon\theta$) condition, where θ is fixed, only the initial condition in (15) changes to $K_0^{(\epsilon)}(y, z) = (1 - 2\epsilon\theta)^{\epsilon^{-1}(z-y)}$, and the initial condition for the limiting PDE (17) changes to $K_0^{(c)}(y, z) = \exp(-2\theta(z - y))$. In the maximal case, $\eta_x = 1$ for all x , the initial conditions $K_0^{(\epsilon)}(y, z) = (-1)^{\epsilon^{-1}(z-y)}$ converge in distribution to the zero function, which is sufficient to imply suitable convergence of the kernels at a fixed $t > 0$.

(b) Branching and coalescing model with maximal initial condition.

We consider the BCRW model with parameters $p_x = q_x = \alpha > 0$ and $l = r = 2\epsilon\sqrt{\alpha\beta} > 0$, and with maximal initial condition, that is $\eta(x) = 1$ for all x . (We have chosen the form of the branching rate parameters so that the limit has a convenient form). From Theorem 3 the process $X_t^{(\epsilon)}$ is a Pfaffian point process on $\epsilon\mathbb{Z}$ with kernel $\mathbf{K}_t^{(\epsilon)}$ of the form

$$\begin{aligned} \mathbf{K}_t^{(\epsilon)}(y, z) &= \epsilon\phi \begin{pmatrix} K_t^{(\epsilon)}(y, z) & -D_2^{(\epsilon)}K_t^{(\epsilon)}(y, z) \\ -D_1^{(\epsilon)}K_t^{(\epsilon)}(y, z) & D_1^{(\epsilon)}D_2^{(\epsilon)}K_t^{(\epsilon)}(y, z) \end{pmatrix} \quad \text{for } y < z, \\ (\mathbf{K}_t^{(\epsilon)})_{12}(y, y) &= 1 - \phi^{-1}K_t^{(\epsilon)}(y, y + \epsilon). \end{aligned} \quad (19)$$

The function $K_t^{(\epsilon)}(y, z)$ solves, for $y, z \in \epsilon\mathbb{Z}$ with $y \leq z$,

$$\begin{cases} \partial_t K_t^{(\epsilon)} &= \alpha\phi\Delta^{(\epsilon)}K_t^{(\epsilon)} - 2\epsilon^{-2}c_0K_t^{(\epsilon)}, \\ K_t^{(\epsilon)}(y, y) &= 1, \\ K_0^{(\epsilon)}(y, z) &= 0. \end{cases} \quad (20)$$

Examination of the constants ϕ and c_0 shows that

$$\phi = 1 + \epsilon\sqrt{\beta/\alpha} - \epsilon^2(\beta/2\alpha) + O(\epsilon^3), \quad c_0 = \beta\epsilon^2 + O(\epsilon^3).$$

The limit X_t is Pfaffian on \mathbb{R} with kernel of the form

$$\begin{aligned} \mathbf{K}_t^{(c)}(y, z) &= \begin{pmatrix} K_t^{(c)}(y, z) & -D_2K_t^{(c)}(y, z) \\ -D_1K_t^{(c)}(y, z) & D_1D_2K_t^{(c)}(y, z) \end{pmatrix} \quad \text{for } y < z, \\ (\mathbf{K}_t^{(c)})_{12}(y, y) &= -D_2K_t^{(c)}(y, y) + \sqrt{\beta/\alpha} \end{aligned} \quad (21)$$

where $K_t^{(c)}(y, z)$ is C^2 on $\{y, z \in \mathbb{R}^2 : y \leq z\}$ and solves

$$\begin{cases} \partial_t K_t^{(c)}(y, z) &= \alpha\Delta K_t^{(c)}(y, z) - 2\beta K_t^{(c)}(y, z) \\ K_t^{(c)}(y, y) &= 1 \\ K_0^{(c)}(y, z) &= 0. \end{cases} \quad (22)$$

Note that the term $(\mathbf{K}_t^{(c)})_{12}(y, y)$ requires a bit more care than in example (a) and that an extra term $\sqrt{\beta/\alpha}$ emerges.

The $t \rightarrow \infty$ limit follows the lines of the previous example, and $X_t^{(c)} \rightarrow X_\infty^{(c)}$ where the limit has Pfaffian kernel that is twice the one in (18), and with the extra difference that

$$\mathbf{K}_{\infty,12}^{(c)}(y, y) = -D_2 K_\infty^{(c)}(y, y) + \sqrt{\frac{\beta}{\alpha}} = 2\sqrt{\frac{\beta}{\alpha}}.$$

Similar row and column manipulations as in the discrete case show that this kernel encodes a Poisson process of rate $2\sqrt{\frac{\beta}{\alpha}}$.

Remarks. 1. In many formulations of determinantal point processes, the determinantal kernel D is associated to an integral operator D on $L^2(\mathbb{R})$, and the diagonal values $D(y, y)$ are linked to those of $(D(y, z) : y < z)$ by the fact that the operator is assumed to be of trace class. One might ask the same for the Pfaffian case, asking for four operators K_{ij} on $L^2(\mathbb{R})$. For our examples this link is broken: the operators K_{ij} acting on $L^2(\mathbb{R})$ would have discontinuities along $y = z$ and are not expected to be trace class. The diagonal values $K_{12}(y, y)$ are not given, for example, as even the continuous limit of $K_{12}(y, z)$. This is also the case for classical Pfaffian kernels, for example for GOE.

The operator formulation is useful, for example when applying the theory of Fredholm determinants or Fredholm Pfaffians, and in classification theorems. However, we state our continuum Pfaffian kernels in the form of the five measurable functions, namely $(\mathbf{K}_{ij}(y, z) : y, z \in \mathbb{R}, y < z)_{i,j \in \{1,2\}}$ and $(\mathbf{K}_{12}(y, y) : y \in \mathbb{R})$. These five functions are what appear in the Pfaffian formulae for the intensities $\rho^{(N)}$. The kernel (21) can be adjusted, by row and column operations, so that for example the diagonal values $K_{12}(y, y)$ are given as the continuous limit of $K_{12}(y, z)$ as $z \downarrow y$, for example to

$$\begin{pmatrix} K_t^{(c)} & -D_2 K_t^{(c)} + \sqrt{\frac{\beta}{\alpha}} K_t^{(c)} \\ -D_1 K_t^{(c)} + \sqrt{\frac{\beta}{\alpha}} K_t^{(c)} & D_1 D_2 K_t^{(c)} - \sqrt{\frac{\beta}{\alpha}} (D_2 K_t^{(c)} + D_1 K_t^{(c)}) + \frac{\beta}{\alpha} K_t^{(c)} \end{pmatrix}$$

This form is more useful for example when manipulating Fredholm Pfaffians (as for example in the manipulations for the gap probabilities for the GOE spectrum).

2. With our convention on kernels just as measurable functions, a Poisson rate γ process can be realised as a Pfaffian point process with kernel $\gamma \mathbf{J}$ where $\mathbf{J}(y, z) = 0$ for $y < z$ and $J_{12}(y, y) = \gamma$. (The same convention would allow Poisson processes to be determinantal processes with a purely diagonal kernel.) The kernel in example (b) is connected to Poisson thickening. A locally finite point process X can be γ *thickened* by adding the points of an independent Poisson process Y of rate γ , producing a new point process $X + Y$. If the original point process was Pfaffian with kernel \mathbf{K} then the thickened process remains Pfaffian with kernel $\mathbf{K} + \gamma \mathbf{J}$. Indeed, since the points of X and the Poisson process never meet, the intensities for the thickened process are given by

$$\rho_{X+Y}^{(N)}(y_1, \dots, y_N) = \sum_{J \subseteq \{1, \dots, N\}} \rho_X^{|J|}(y_j : j \in J) \gamma^{N-|J|}$$

where $|J|$ is the size of the subset J . But the Pfaffian $\text{Pf}(\mathbf{K} + \gamma\mathbf{J})$ can be expanded by the Pfaffian sum formula (see appendix of) to give exactly this relation.

A locally finite point process X can be γ *thinned* by removing each point independently with probability γ , producing a new point process which we denote $\Theta_\gamma(X)$. If X is Pfaffian with kernel \mathbf{K} then the thinned process $\Theta_\gamma(X)$ remains Pfaffian, with the kernel $\gamma\mathbf{K}$.

Since the PDE behind both the continuum branching process and the continuum pairwise immigration model is the same, the heat equation with constant potential, it is not surprising that there is a connection between their fixed time distribution. Using thickening and thinning we can state this: let

$$\begin{aligned} X_1 &= \text{the diffusion limit of BCRW with } p = q = \alpha, l = r = 2\epsilon\sqrt{\alpha\beta} \text{ and } \eta \equiv 1; \\ X_2 &= \text{the diffusion limit of ARWPI with } p = q = \alpha, m = \epsilon^2\beta \text{ and } \eta \equiv 1; \\ Y &= \text{a Poisson point process of rate } \frac{1}{2}\sqrt{\beta/\alpha}, \text{ independent of } X_2. \end{aligned}$$

Then, as point processes on \mathbb{R} ,

$$\Theta_{1/2}(X_1) \stackrel{\mathcal{D}}{=} X_2 + Y.$$

The proof is just the verification that the Pfaffian kernels are identical.

There is a nice dynamic coupling argument that connects annihilating Brownian motions with coalescing Brownian motions (see [13]), but we do not know a dynamic coupling that explains the above equality of distributions.

A similar identity also works for carefully chosen Poisson initial conditions, and also for the processes on \mathbb{Z} with suitably chosen initial conditions (see details in the thesis by Garrod [4]).

(c) Branching and coalescing model with a single initial particle.

We take the same parameter choices as in example (a) but choose an initial condition that is a single particle at the origin. The initial conditions for (20) and (21) change to

$$K_0^{(\epsilon)}(y, z) = \phi^{z-y} \mathbf{I}(0 \notin [y, z]), \quad K_0^{(c)}(y, z) = e^{\sqrt{\frac{\beta}{\alpha}}(z-y)} \mathbf{I}(0 \notin [y, z]).$$

The explicit solution is

$$K_t^{(c)}(y, z) = e^{\sqrt{\frac{\beta}{\alpha}}(z-y)} (1 - \psi_t(y)\psi_t(-z)) + e^{-\sqrt{\frac{\beta}{\alpha}}(z-y)} \psi_t(-y)\psi_t(z), \quad (23)$$

where

$$\psi_t(x) = 2 \operatorname{erfc} \left(\frac{x - 2\sqrt{\alpha\beta}t}{\sqrt{2\alpha t}} \right).$$

As in the discrete setting, the fixed time distribution started from a single site is quite easy to understand. The limit behaviour of the leftmost and rightmost particles $\{l_t, r_t\}$, under the parameter scaling we have used, is known to become that of a sticky pair $\{L_t, R_t\}$, that is the solution of

$$\begin{aligned} dL_t &= \mathbf{I}(L_t \neq R_t) dB_t^L + \mathbf{I}(L_t = R_t) dB - 2\sqrt{\alpha\beta}dt, \quad L_0 = 0, \\ dR_t &= \mathbf{I}(L_t \neq R_t) dB_t^R + \mathbf{I}(L_t = R_t) dB + 2\sqrt{\alpha\beta}dt, \quad R_0 = 0, \end{aligned}$$

where B^R, B^L, B are three independent rate 2α Brownian motions. Uniqueness in law holds and $L_t \leq R_t$ for all $t \geq 0$. Let Y be an independent Poisson process of rate $2\sqrt{\frac{\beta}{\alpha}}$. The point process $X_t^{(c)}$ can be constructed as the pair of particles L_t and R_t together with the particles from Y that lie inside (L_t, R_t) . Indeed then

$$\mathbb{P}[X_t[y, z] = 0] = \mathbb{P}[R_t < y] + \mathbb{P}[L_t \geq z] + e^{-2\sqrt{\frac{\beta}{\alpha}}(z-y)} \mathbb{P}[L_t < y, R_t \geq z].$$

Comparing this with the formula (23) for $K_t^{(c)}(y, z) = e^{\sqrt{\frac{\beta}{\alpha}}(z-y)} \mathbb{P}[X_t[y, z] = 0]$, and using $\psi_t(x) = \mathbb{P}[R_t \geq x] = \mathbb{P}[L_t < -x]$, one finds that

$$\mathbb{P}[L_t < y, R_t \geq z] = \mathbb{P}[L_t < y] \mathbb{P}[R_t \geq z] - e^{2\sqrt{\frac{\beta}{\alpha}}(z-y)} \mathbb{P}[L_t \geq z] \mathbb{P}[R_t < y].$$

This formula, which is straightforward to verify independently, is one way of describing the joint law of (L_t, R_t) .

(d) Annihilating model with immigration at the origin.

We allow immigration only at one site, namely the origin, producing a model we have come to call the Brownian firework. The immigration rate must be scaled differently to example (c) in order to see a non-trivial effect in the continuum limit. Thus we consider the ARWPI model with parameters $p_x = q_x = \alpha > 0$ for all x , with $m_0 = \beta\epsilon^{-1} \geq 0$ and $m_x = 0$ for all $x \neq 0$, and with zero initial condition. From Theorem 3 the process $X_t^{(\epsilon)}$ is a Pfaffian point process on $\epsilon\mathbb{Z}$ with kernel $\mathbf{K}_t^{(\epsilon)}$ of the form

$$\mathbf{K}_t^{(\epsilon)}(y, z) = \frac{\epsilon}{2} \begin{pmatrix} K_t^{(\epsilon)}(y, z) & -D_2^{(\epsilon)} K_t^{(\epsilon)}(y, z) \\ -D_1^{(\epsilon)} K_t^{(\epsilon)}(y, z) & D_1^{(\epsilon)} D_2^{(\epsilon)} K_t^{(\epsilon)}(y, z) \end{pmatrix} \quad \text{for } y < z, \quad (24)$$

and $(\mathbf{K}_t^{(\epsilon)})_{12}(y, y) = -\frac{\epsilon}{2} D_2^{(\epsilon)} K_t^{(\epsilon)}(y, y)$, where the function $K_t^{(\epsilon)}(y, z)$ solves, for $y, z \in \epsilon\mathbb{Z}$ with $y \leq z$,

$$\begin{cases} \partial_t K_t^{(\epsilon)} &= \alpha \Delta^{(\epsilon)} K_t^{(\epsilon)} - 2\beta\epsilon^{-1}(\mathbf{I}(y=0) + \mathbf{I}(z=0))K_t^{(\epsilon)}, \\ K_t^{(\epsilon)}(y, y) &= 1, \\ K_0^{(\epsilon)}(y, z) &= 1. \end{cases}$$

The limiting kernel $K_t^{(c)}(y, z)$ solves

$$\begin{cases} \partial_t K_t^{(c)}(y, z) &= \alpha \Delta K_t^{(c)}(y, z) - 2\beta(\delta_{y=0} + \delta_{z=0})K_t^{(c)}(y, z) \\ K_t^{(c)}(y, y) &= 1 \\ K_0^{(c)}(y, z) &= 1. \end{cases}$$

This limiting PDE has a distribution potential consisting of delta functions on the $y = 0$ and $z = 0$ axes. However it has unique bounded continuous mild solutions, which are smooth away from the axes. We first show convergence $K_t^{(\epsilon)}$. The probabilistic representation of the limiting continuous PDE is

$$K_t^{(c)}(y, z) = \mathbb{E} \left[e^{-\frac{\beta}{\alpha} L_t^Y - \frac{\beta}{\alpha} L_t^Z} \right]$$

where L^Y and L^Z are the (semimartingale) local times of Y and Z at zero. The corresponding formula

$$K_t^{(\epsilon)}(y, z) = \mathbb{E} \left[e^{-\frac{\beta}{\alpha} L_{t \wedge \tau}^{Y^{(\epsilon)}} - \frac{\beta}{\alpha} L_{t \wedge \tau}^{Z^{(\epsilon)}}} \right]$$

holds for random walks $Y^{(\epsilon)}, Z^{(\epsilon)}$ on $\epsilon\mathbb{Z}$, jumping right and left each with rate $\epsilon^{-2}\alpha$, and their local times, for example

$$L_t^{Y^{(\epsilon)}} = 2\alpha\epsilon^{-1} \int_0^t \mathbf{I}(Y_s^{(\epsilon)} = 0) ds.$$

The weak convergence of $(Y^{(\epsilon)}, Z^{(\epsilon)}, L^{Y^{(\epsilon)}}, L^{Z^{(\epsilon)}}) \rightarrow (Y, Z, L^Y, L^X)$ can be used to check that

$$K_t^{(\epsilon)}(y_\epsilon, z_\epsilon) \rightarrow K_t^{(c)}(y, z) \quad \text{whenever } y_\epsilon \rightarrow y, z_\epsilon \rightarrow z.$$

The (complicated) explicit formula for $K_t^{(c)}(y, z)$ reveals that the intensity

$$\rho_t^{(1)}(y) = -D_2 K_t^{(c)}(y, y+) \uparrow \infty \quad \text{as } y \rightarrow 0.$$

Thus the boundedness conditions in the kernel convergence lemma (Lemma 4 from [3]) can never hold. We may however break the equation (25) into three heat equations, on $\{y < z < 0\}$, on $\{y < 0, z > 0\}$ and on $\{0 < y < z\}$, each with Dirichlet boundary conditions given by the values of $K^{(c)}$. These equations have C^2 solutions on their domains at any fixed time, and the lattice approximations converge suitably (that is they and their derivatives are bounded and converge uniformly) provided one stays away from the boundaries. We choose $\delta > 0$ and consider the restriction of the process on $\mathbb{R} \setminus (-\delta, \delta)$. Then we can consider the function $K_t^{(\epsilon)}(y, z)$ as a lattice approximation to the simple heat equation on $(\mathbb{R} \setminus (-\delta, \delta))^2$ and the kernel convergence lemma allows us to construct a limiting Pfaffian point process on $\mathbb{R} \setminus (-\delta, \delta)$. Then by consistency we can take $\delta \downarrow 0$ and construct the limiting point process on $\mathbb{R} \setminus \{0\}$.

Using the explicit joint law for (X_t, L_t^X) one can solve explicitly for $K_t^{(c)}(y, z)$. Using this one can check there are limits as $t \rightarrow \infty$ for $K_t^{(c)}$ and its first two derivatives as $t \rightarrow \infty$. Indeed we find

$$K_\infty^{(c)}(y, z) = 1 + \frac{2\beta}{\pi\alpha} \int_0^\infty e^{-\frac{\beta}{\alpha}s} \left(\arctan \frac{y}{s+|z|} - \arctan \frac{z}{s+|y|} \right) ds.$$

Again $X_t^{(c)} \rightarrow X_\infty^{(c)}$ when restricted to $\mathbb{R} \setminus (-\delta, \delta)$. A further limit can be taken as the immigration rate at the origin β increases to an infinite rate, and this yields the kernel (7) stated in the introduction.

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